Local and global Casimir energies for a semitransparent cylindrical shell

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Abstract

The local Casimir energy density and the global Casimir energy for a massless scalar field associated with a $\lambda\delta$-function potential in a $(3 + 1)$-dimensional circular cylindrical geometry are considered. The global energy is examined for both weak and strong coupling, the latter being the well-studied Dirichlet cylinder case. For weak coupling, through $\mathcal{O}(\lambda^2)$, the total energy is shown to vanish by both analytic and numerical arguments, based both on Green's-function and zeta-function techniques. Divergences occurring in the calculation are shown to be absorbable by renormalization of physical parameters of the model. The global energy may be obtained by integrating the local energy density only when the latter is supplemented by an energy term residing precisely on the surface of the cylinder. The latter is identified as the integrated local energy density of the cylindrical shell when the latter is physically expanded to have finite thickness. Inside and outside the δ-function shell, the local energy density diverges as the surface of the shell is approached; the divergence is weakest when the conformal stress tensor is used to define the energy density. A real global divergence first occurs in $\mathcal{O}(\lambda^3)$, as anticipated, but the proof is supplied here for the first time; this divergence is entirely associated with the surface energy and does *not* reflect divergences in the local energy density as the surface is approached.

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1. Introduction

The subject of self-energies due to quantum fluctuations of fields constrained by physical boundaries, such as the Casimir energy due to a perfectly conducting spherical shell [1], has been controversial nearly from the outset [2, 3]. Partly this is because such a self-energy is

apparently not well defined, for it is not accessible by deforming part of the surface. More fundamentally, it is because there are strong divergences in the local energy density as the (idealized) boundary is approached, which would seem to rule out the existence of a finite total energy for the system. How are such divergences to be squared with the mathematical proof that the electromagnetic Casimir energy of a region with a closed, smooth, perfectly conducting boundary is finite [4]?

These issues have been brought to the forefront by a series of papers by Graham *et al* [5]. Essentially, they assert that it is impossible to ascribe any physical meaning to the self-Casimir energy of a single object, such as Boyer's sphere [1]. However, the divergence issues they raise are hardly new [6]; for example, for a massless scalar field in the presence of a spherical δ-shell potential, governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\frac{\lambda}{a^2}\delta(r-a)\phi^2, \tag{1.1}$$

a divergence occurs in third order in λ [7], and possible ways of dealing with it have been suggested [8, 9].

For a different viewpoint see the considerations of Barton [10]. He argues that for any *single connected body* the physics is dominated by the divergent components of the self-energy. The pure Casimir terms, which in the examples studied in [10] are the convergent components when the no-cutoff limit is taken, are shown to be much smaller such that they will never become observable.

In a recent paper [11], the model described by (1.1) was considered in some detail. In fact, the model examined was somewhat more general, in that the δ function was replaced by a step-function potential of width δ and height h. As $\delta \to 0$ and $h \to \infty$ with $h\delta$ held fixed at unity, a δ-function potential is recovered. For such a potential we calculated the energy density inside and outside the region of the potential, from which we could calculate the total energy as a function of λ, as well as the local energy density. As long as the potential is finite, the local energy density may be integrated to yield the total energy. In the singular δ-function limit, however, the energy within the shell becomes a localized surface energy which must be added to the integrated local energy of the regions inside $(r < a)$ and outside $(r > a)$ the sphere, which has precisely the anticipated form [12–15]. For weak coupling with the δ-shell potential, the total Casimir energy is finite in $\mathcal{O}(\lambda^2)$, but divergent in third order, which divergence precisely corresponds to the divergence of the surface energy in that order. Thus, it is plausible that such a divergence should be absorbed in a renormalization of the surface energy.

In this paper, we turn to the corresponding cylindrical case. The situation in many ways is similar. However, there are curiosities associated with the cylindrical geometry that make this new analysis intriguing. First, the Casimir self-stress on a perfectly conducting circular cylinder was found [16] to be attractive, and of somewhat smaller magnitude, compared to the repulsive stress found by Boyer for a perfectly conducting sphere [1]. It was found that a dilute dielectric cylinder had vanishing van der Waals energy [17, 18], which seemed to imply the vanishing of the Casimir energy for a dielectric cylinder of permittivity ε in order $(\varepsilon - 1)^2$, as was only recently verified [19, 20]. The same was true for a dilute dielectric-diamagnetic cylinder (with the speed of light the same on the inside and the outside) [18, 21, 22]. If a perfectly conducting cylinder is slightly deformed by giving its cross section a slight eccentricity δe, the change in the Casimir energy vanishes in order δe^2 [23]. None of these vanishings occur for a sphere [24–29], so they all reflect the flatness of the cylindrical geometry. In this paper, we establish another example of the second-order vanishing effect for the cylinder, that is, that the $\mathcal{O}(\lambda^2)$ term in the Casimir energy for the semitransparent cylinder

Table 1. Casimir energy (E) for a sphere and Casimir energy per unit length ($\mathcal{E}$) for a cylinder, both of radius a. Here, the different boundary conditions are perfectly conducting for electromagnetic fields (EM), Dirichlet for scalar fields (D), dilute dielectric for electromagnetic fields (coefficient of $(\varepsilon - 1)^2$), dilute dielectric for electromagnetic fields with media having the same speed of light (coefficient of $\xi^2 = [(\varepsilon - 1)/(\varepsilon + 1)]^2$), for perfectly conducting surface with eccentricity δe (coefficient of δe^2) and weak coupling for scalar field with δ-function boundary given by (1.1), (coefficient of λ^2). The references given are, to the authors' knowledge, the first paper in which the results in the various cases were found.

Type	$E_{\text{Sphere}}a$	$\mathcal{E}_{\text{Cylinder}}a^2$	References
EM	+0.046 18	−0.013 56	[1, 16]
D	+0.002 817	+0.000 6148	[30, 31]
$(\varepsilon - 1)^2$	+0.004 767 = $\frac{23}{1536\pi}$	0	[25, 19]
ξ^2	+0.049 74 = $\frac{5}{32\pi}$	0	[28, 18]
δe^2	±0.000 9	0	[29, 23]
λ^2	+0.009 947 = $\frac{1}{32\pi}$	0	[32]

is zero. A summary of the facts comparing results for sphere and cylinder, together with the first reference for each result, is given in table 1.

The outline of this paper is as follows. In the next section, we will derive the Green's function for the semitransparent cylinder. In section 3, we will then compute the Casimir pressure on the cylinder, and thereby infer the Casimir energy, which we can also derive directly. The weak-coupling evaluation of the Casimir energy will be the subject of section 4. We will show that the Casimir energy through order λ^2 vanishes both by analytic and numerical arguments, with an explicit isolation of the divergent term which may be unambiguously removed. An independent derivation of these results using zeta-function techniques is given in section 5. The proof that for a cylinder of arbitrary cross section the divergence in the Casimir energy occurs in $\mathcal{O}(\lambda^3)$ is supplied in section 6. As we have noted in table 1, the strong-coupling (Dirichlet) result was earlier derived by Gosdzinsky and Romeo [31]; we reproduce their result in section 7. Again the explicit divergent terms that must be removed by renormalization are identified. We turn to an examination of the local energy density in section 8. We show that the integrated local energy density differs from the total energy by a surface term that resides precisely on the cylindrical surface. We also examine the surface divergences that appear in the energy density as one approaches the surface; the leading divergence, which is independent of the shape of the surface, may be eliminated by choosing the conformal stress tensor. Then one is left with surface divergences, in either strong or weak coupling, that are exactly one-half that for a sphere, reflecting the vanishing of one of the principal curvatures for a cylinder. In section 9, we examine what happens when the δ-function potential is thickened to a cylindrical annulus of thickness δ and height h. We show that then the surface energy is resolved as the integrated local energy density of the field within the confines of the annulus. The divergence in the total energy in $\mathcal{O}(\lambda^3)$ for a δ-function shell is exactly that due to the surface energy alone. Finally, we offer a perspective of the situation in the conclusion. Appendix A offers further details on the divergences occurring in the global theory, particularly in $\mathcal{O}(\lambda)$, while appendix B elaborates some further aspects of the surface energy.

2. Green's function

We consider a massless scalar field ϕ in a δ-cylinder background,

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{2a}\delta(r - a)\phi^2, \tag{2.1}$$

with a being the radius of the 'semitransparent' cylinder. We recall that the massive case was earlier considered by Scandurra [33]. Note that with this definition, λ is dimensionless. We will assume that the coupling $\lambda > 0$ to avoid the appearance of negative eigenfrequencies. The time-Fourier transform of the Green's function,

$$G(x, x') = \int \frac{d\omega}{2\pi} \, e^{-i\omega(t-t')} \mathcal{G}(\mathbf{r}, \mathbf{r}'),$$ (2.2)

satisfies

$$\left[-\nabla^2 - \omega^2 + \frac{\lambda}{a} \delta(r - a) \right] \mathcal{G}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$ (2.3)

Adopting cylindrical coordinates, we write

$$\mathcal{G}(\mathbf{r}, \mathbf{r}') = \int \frac{dk}{2\pi} \, e^{ik(z-z')} \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{im(\varphi - \varphi')} g_m(r, r'; k),$$ (2.4)

where the reduced Green's function satisfies

$$\left[-\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + \kappa^2 + \frac{m^2}{r^2} + \frac{\lambda}{a} \delta(r - a) \right] g_m(r, r'; k) = \frac{1}{r} \delta(r - r'),$$ (2.5)

where $\kappa^2 = k^2 - \omega^2$. Let us immediately make a Euclidean rotation,

$$\omega \to i\zeta,$$ (2.6)

where ζ is real, so κ is likewise always real. Apart from the δ functions, this is the modified Bessel equation.

Because of the Wronskian satisfied by the modified Bessel functions, we have the general solution to (2.5) as long as $r \neq a$ to be

$$g_m(r, r'; k) = I_m(\kappa r_<) K_m(\kappa r_>) + A(r') I_m(\kappa r) + B(r') K_m(\kappa r),$$ (2.7)

where A and B are arbitrary functions of r'. Now we incorporate the effect of the δ function at $r = a$ in (2.5). It implies that g_m must be continuous at $r = a$, while it has a discontinuous derivative,

$$a \frac{d}{dr} g_m(r, r'; k) \bigg|_{r=a-}^{r=a+} = \lambda g_m(a, r'; k),$$ (2.8)

from which we rather immediately deduce the form of the Green's function inside and outside the cylinder:

$$r, r' < a: \quad g_m(r, r'; k) = I_m(\kappa r_<) K_m(\kappa r_>) - \frac{\lambda K_m^2(\kappa a)}{1 + \lambda I_m(\kappa a) K_m(\kappa a)} I_m(\kappa r) I_m(\kappa r'), \quad (2.9a)$$

$$r, r' > a: \quad g_m(r, r'; k) = I_m(\kappa r_<) K_m(\kappa r_>) - \frac{\lambda I_m^2(\kappa a)}{1 + \lambda I_m(\kappa a) K_m(\kappa a)} K_m(\kappa r) K_m(\kappa r'). \quad (2.9b)$$

Note that in the limit $\lambda \to \infty$ we recover the Dirichlet cylinder result, that is, that g_m vanishes at $r = a$.

3. Pressure and energy

The easiest way to calculate the total energy is to compute the pressure on the cylindrical walls due to the quantum fluctuations in the field. This may be computed, at the one-loop level, from the vacuum expectation value of the stress tensor,

$$\langle T^{\mu\nu} \rangle = \left(\partial^\mu \partial'^\nu - \frac{1}{2} g^{\mu\nu} \partial^\lambda \partial'_\lambda \right) \frac{1}{i} G(x, x') \bigg|_{x=x'} - \xi(\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \frac{1}{i} G(x, x),$$ (3.1)

which we have written in a Cartesian coordinate system. Here, we have included the conformal parameter ξ, which is equal to $1/6$ for the stress tensor that makes conformal invariance manifest. The conformal term does not contribute to the radial–radial component of the stress tensor, however, because then only transverse and time derivatives act on $G(x, x)$, which depends only on r. The discontinuity of the expectation value of the radial–radial component of the stress tensor is the pressure of the cylindrical wall:

$$
\begin{aligned}
P &= \langle T_{rr} \rangle_{\text{in}} - \langle T_{rr} \rangle_{\text{out}} \\
&= -\frac{1}{16\pi^3} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}\zeta \frac{\lambda \kappa^2}{1 + \lambda I_m(\kappa a) K_m(\kappa a)} \\
&\quad \times \left[K_m^2(\kappa a) I_m'^2(\kappa a) - I_m^2(\kappa a) K_m'^2(\kappa a) \right] \\
&= -\frac{1}{16\pi^3} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}\zeta \frac{\kappa}{a} \frac{\mathrm{d}}{\mathrm{d}\kappa a} \ln[1 + \lambda I_m(\kappa a) K_m(\kappa a)], \quad (3.2)
\end{aligned}
$$

where we have again used the Wronskian. Regarding ka and ζa as the two Cartesian components of a two-dimensional vector, with magnitude $x \equiv \kappa a = \sqrt{k^2 a^2 + \zeta^2 a^2}$, we get the stress on the cylinder per unit length to be

$$
S = 2\pi a P = -\frac{1}{4\pi a^3} \int_0^\infty \mathrm{d}x\, x^2 \sum_{m=-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \ln[1 + \lambda I_m(x) K_m(x)], \quad (3.3)
$$

which possesses the expected Dirichlet limit as $\lambda \to \infty$. The corresponding expression for the total Casimir energy per unit length follows by integrating

$$
S = -\frac{\partial}{\partial a}\mathcal{E}, \quad (3.4)
$$

that is,

$$
\mathcal{E} = -\frac{1}{8\pi a^2} \int_0^\infty \mathrm{d}x\, x^2 \sum_{m=-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \ln[1 + \lambda I_m(x) K_m(x)]. \quad (3.5)
$$

This expression is of course completely formal and will be regulated in various ways in the following, for example, with an exponential regulator (in sections 4 and 7, appendix A) or by using zeta-function regularization (in section 5).

Alternatively, we may compute the energy directly from the general formula [34]

$$
E = \frac{1}{2\mathrm{i}} \int (\mathrm{d}\mathbf{r}) \int \frac{\mathrm{d}\omega}{2\pi} 2\omega^2 \mathcal{G}(\mathbf{r}, \mathbf{r}). \quad (3.6)
$$

To evaluate (3.6) in this case, we use the standard indefinite integrals over squared Bessel functions. When we insert the above construction of the Green's function and perform the integrals as indicated over the regions interior and exterior to the cylinder we obtain

$$
\mathcal{E} = -\frac{a^2}{8\pi^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\zeta \int_{-\infty}^{\infty} \mathrm{d}k\, \zeta^2 \frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x} \ln[1 + \lambda I_m(x) K_m(x)]. \quad (3.7)
$$

Again we regard the two integrals as over Cartesian coordinates, and the result (3.5) immediately follows.

4. Weak-coupling evaluation

Suppose we regard λ as a small parameter, so let us expand (3.5) in powers of λ. The first term is

$$\mathcal{E}^{(1)} = -\frac{\lambda}{8\pi a^2} \sum_{m=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}x \, x^2 \frac{\mathrm{d}}{\mathrm{d}x} K_m(x) I_m(x). \tag{4.1}$$

The addition theorem for the modified Bessel functions is

$$K_0(kP) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')} K_m(k\rho) I_m(k\rho'), \qquad \rho > \rho', \tag{4.2}$$

where $P = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}$. If this is extrapolated to the limit $\rho' = \rho$ we conclude that the sum of the Bessel functions appearing in (4.1) is $K_0(0)$, that is, a constant, so there is no first-order contribution to the energy. For a rigorous derivation of this result, see section 5.1 and also appendix A.

4.1. Analytic regularization

We can proceed the same way to evaluate the second-order contribution,

$$\mathcal{E}^{(2)} = \frac{\lambda^2}{16\pi a^2} \int_0^{\infty} \mathrm{d}x \, x^2 \frac{\mathrm{d}}{\mathrm{d}x} \sum_{m=-\infty}^{\infty} I_m^2(x) K_m^2(x). \tag{4.3}$$

By squaring the sum rule (4.2), and taking the limit $\rho' \to \rho$, we evaluate the sum over Bessel functions appearing here as

$$\sum_{m=-\infty}^{\infty} I_m^2(x) K_m^2(x) = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} K_0^2(2x \sin \varphi/2). \tag{4.4}$$

Then changing the order of integration, the second-order energy can be written as

$$\mathcal{E}^{(2)} = -\frac{\lambda^2}{64\pi^2 a^2} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\sin^2 \varphi/2} \int_0^{\infty} \mathrm{d}z \, z K_0^2(z), \tag{4.5}$$

where the Bessel-function integral has the value $1/2$. However, the integral over φ is divergent. We interpret this integral by adopting an analytic regularization based on the integral [19]

$$\int_0^{2\pi} \mathrm{d}\varphi \left(\sin \frac{\varphi}{2} \right)^s = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1 + \frac{s}{2}\right)}, \tag{4.6}$$

which holds for $\mathrm{Re}\, s > -1$. Taking the right-hand side of this equation to define the φ integral for all s, we conclude that the φ integral in (4.5), and hence the second-order energy $\mathcal{E}^{(2)}$, is zero.

4.2. Numerical evaluation

Given that the above argument evidently formally omits divergent terms, it may be more satisfactory, as in [19], to offer a numerical evaluation of $\mathcal{E}^{(2)}$. (The corresponding argument for $\mathcal{E}^{(1)}$ is given in appendix A.) We can very efficiently do so using the uniform asymptotic expansions ($m \to \infty$):

$$I_m(x) \sim \sqrt{\frac{t}{2\pi m}} \, \mathrm{e}^{m\eta} \left(1 + \sum_k \frac{u_k(t)}{m^k} \right), \tag{4.7a}$$

$$K_m(x) \sim \sqrt{\frac{\pi t}{2m}} \, e^{-m\eta} \left(1 + \sum_k (-1)^k \frac{u_k(t)}{m^k} \right), \tag{4.7b}$$

where $x = mz, t = 1/\sqrt{1 + z^2}$, and the value of η is irrelevant here. The polynomials in t appearing in (4.7) are generated by

$$u_0(t) = 1, \qquad u_k(t) = \frac{1}{2} t^2 (1 - t^2) u'_{k-1}(t) + \frac{1}{8} \int_0^t ds (1 - 5s^2) u_{k-1}(s). \tag{4.8}$$

Thus, the asymptotic behaviour of the product of Bessel functions appearing in (4.3) is

$$I_m^2(x) K_m^2(x) \sim \frac{t^2}{4m^2} \left(1 + \sum_{k=1}^{\infty} \frac{r_k(t)}{m^{2k}} \right). \tag{4.9}$$

The first three polynomials occurring here are

$$r_1(t) = \frac{t^2}{4} (1 - 6t^2 + 5t^4), \tag{4.10a}$$

$$r_2(t) = \frac{t^4}{16} (7 - 148t^2 + 554t^4 - 708t^6 + 295t^8), \tag{4.10b}$$

$$r_3(t) = \frac{t^6}{16} (36 - 1666t^2 + 13\,775t^4 - 44\,272t^6 + 67\,162t^8 - 48\,510t^{10} + 13\,475t^{12}). \tag{4.10c}$$

We now write the second-order energy (4.3) as

$$\mathcal{E}^{(2)} = -\frac{\lambda^2}{8\pi a^2} \left\{ \int_0^{\infty} dx \, x \left[I_0^2(x) K_0^2(x) - \frac{1}{4(1 + x^2)} \right] - \frac{1}{4} \lim_{s \to 0} \left(\frac{1}{2} + \sum_{m=1}^{\infty} m^{-s} \right) \right.$$

$$\times \int_0^{\infty} dz \, z^{2-s} \frac{d}{dz} \frac{1}{1 + z^2} + 2 \int_0^{\infty} dz \, z \frac{t^2}{4} \sum_{m=1}^{\infty} \sum_{k=1}^{3} \frac{r_k(t)}{m^{2k}}$$

$$\left. + 2 \sum_{m=1}^{\infty} \int_0^{\infty} dx \, x \left[I_m^2(x) K_m^2(x) - \frac{t^2}{4m^2} \left(1 + \sum_{k=1}^{3} \frac{r_k(t)}{m^{2k}} \right) \right] \right\}. \tag{4.11}$$

In the final integral $z = x/m$. The successive terms are evaluated as

$$\mathcal{E}^{(2)} \approx -\frac{\lambda^2}{8\pi a^2} \left[\frac{1}{4} (\gamma + \ln 4) - \frac{1}{4} \ln 2\pi - \frac{\zeta(2)}{48} + \frac{7\zeta(4)}{1920} - \frac{31\zeta(6)}{16\,128} \right.$$

$$\left. + 0.000\,864 + 0.000\,006 \right] = -\frac{\lambda^2}{8\pi a^2} (0.000\,000), \tag{4.12}$$

where in the last term in (4.11) only the $m = 1$ and 2 terms are significant. Therefore, we have demonstrated numerically that the energy in order λ^2 is zero to an accuracy of better than 10^{-6}.

The astute reader will note that we used a standard, but possibly questionable, analytic regularization in defining the second term in (4.11), where the initial sum and integral are only defined for $1 < s < 2$, and then the result is continued to $s = 0$. Alternatively, we could follow [19] and insert there an exponential regulator in each integral of $e^{-x\delta}$, with δ to be taken to zero at the end of the calculation. For $m \neq 0$ x becomes mz, and then the sum on m becomes

$$\sum_{m=1}^{\infty} e^{-mz\delta} = \frac{1}{e^{z\delta} - 1}. \tag{4.13}$$

Then when we carry out the integral over z we obtain for that term

$$\frac{\pi}{8\delta} - \frac{1}{4} \ln 2\pi. \tag{4.14}$$

Thus, we obtain the same finite part as above, but in addition an explicitly divergent term

$$\mathcal{E}_{\mathrm{div}}^{(2)} = -\frac{\lambda^2}{64a^2\delta}. \tag{4.15}$$

If we think of the cutoff in terms of a vanishing proper time τ, $\delta = \tau/a$, this divergent term is proportional to $1/a$, so the divergence in the energy goes like L/a, if L is the (very large) length of the cylinder. This is of the form of the shape divergence encountered in [19].

5. Zeta-function approach

For the massless case, in the zeta-function scheme the regularized total energy reads

$$\mathcal{E} = -\frac{1}{8\pi a^2} a^{2s} (1 + s[-1 + 2\ln(2\mu)]) \sum_{m=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}x \, x^{2-2s} \frac{\mathrm{d}}{\mathrm{d}x} \ln(1 + \lambda K_m(x) I_m(x)), \tag{5.1}$$

where μ is an arbitrary mass scale. This result can, for example, be taken from [33], see also (8.3.16) of [35]. In order to find the total energy one needs to find the analytic continuation of this expression to $s = 0$. Formally, in the $s \to 0$ limit, this is exactly (3.5), so this is the zeta-function regularization of that result.

5.1. $\mathcal{E}^{(1)}$

The first order of the energy, in the weak-coupling expansion, reads

$$\begin{aligned}
\mathcal{E}^{(1)} &= -\frac{\lambda}{8\pi a^2} a^{2s} (1 + s[-1 + 2\ln(2\mu)]) \sum_{m=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}x \, x^{2-2s} \frac{\mathrm{d}}{\mathrm{d}x} K_m(x) I_m(x) \\
&= -\frac{\lambda}{8\pi a^2} a^{2s} (1 + s[-1 + 2\ln(2\mu)]) \left\{ \int_0^{\infty} \mathrm{d}x \, x^{2-2s} \frac{\mathrm{d}}{\mathrm{d}x} K_0(x) I_0(x) \right. \\
&\quad \left. + 2 \sum_{m=1}^{\infty} m^{2-2s} \int_0^{\infty} \mathrm{d}z \, z^{2-2s} \frac{\mathrm{d}}{\mathrm{d}z} K_m(mz) I_m(mz) \right\},
\end{aligned} \tag{5.2}$$

where, for the purpose of the following argument, the $m = 0$ contribution has been separated off. To be precise, let us mention that the $m = 0$ contribution should be thought of as coming from a limit $M \to 0$ of the integral $\int_M^{\infty}$. This is what one actually obtains in the contour integral formalism, for full details see [35]. With this in mind, (5.2) is analytic in the strip $1 < \mathrm{Re}\, s < 3/2$. The substitution $x = mz$ has been done in the last integral to make the convergence properties of the series and the integral better apparent when using the uniform asymptotics of the Bessel functions (as noted above; see below).

We want to find the analytic continuation of this expression to $s = 0$. As it stands, the expression is not valid about $s = 0$ and manipulations need to be done. In order to evaluate it further, one would like to interchange summation and integration and use the addition theorem for the Bessel functions. From the uniform asymptotic expansion of the Bessel functions ($m > 0$),

$$K_m(mz) I_m(mz) \sim \frac{t}{2m} + \frac{t^3}{16m^3} (1 - 6t^2 + 5t^4) + \mathcal{O}\left(\frac{1}{m^5}\right), \tag{5.3}$$

as we have seen already in (4.9) and (4.10a), it is clear that the resulting summation is divergent at $s = 0$; one can see that as it stands the summation is convergent only for $\text{Re } s > 1$. One therefore needs to subtract and add as many asymptotic terms as needed to make the resulting summation convergent. In this procedure, the hope is always that the asymptotic terms which potentially could contain singular contributions at $s = 0$ can be handled analytically. Subtracting all asymptotic terms given above, one sees the resulting summation is well defined at $s = 0$.

Let us define $\tilde{\mathcal{E}}^{(1)}$ without the prefactor in (5.2),

$$\tilde{\mathcal{E}}^{(1)} = \sum_{m=-\infty}^{\infty} \int_0^\infty dx \, x^{2-2s} \frac{d}{dx} K_m(x) I_m(x), \tag{5.4}$$

which, given the above remarks, we rewrite as

$$\tilde{\mathcal{E}}^{(1)} = \int_0^\infty dx \, x^{2-2s} \frac{d}{dx} K_0(x) I_0(x) + 2 \sum_{m=1}^{\infty} m^{2-2s} \int_0^\infty dz \, z^{2-2s} \frac{d}{dz} \left[\frac{t}{2m} + \frac{t^3(1 - 6t^2 + 5t^4)}{16m^3} \right]$$

$$+ 2 \sum_{m=1}^{\infty} m^{2-2s} \int_0^\infty dz \, z^{2-2s} \frac{d}{dz} \left[K_m(mz) I_m(mz) - \frac{t}{2m} - \frac{t^3(1 - 6t^2 + 5t^4)}{16m^3} \right]. \tag{5.5}$$

We need to find the analytical continuation of this expression to $s = 0$. For the first term we find for $\frac{1}{2} < \text{Re } s < 1$

$$\int_0^\infty dx \, x^{2-2s} \frac{d}{dx} K_0(x) I_0(x) = -\frac{\Gamma(2 - s)\Gamma(1 - s)\Gamma\left(s - \frac{1}{2}\right)}{2\sqrt{\pi}\,\Gamma(s)}, \tag{5.6}$$

where the right-hand side vanishes when analytically continued to $s = 0$.

For the middle, asymptotic, terms in (5.5) we use for $1 - \frac{n}{2} < \text{Re } s < 2$

$$\int_0^\infty dz \, z^{2-2s} \frac{d}{dz} \frac{1}{(1 + z^2)^{n/2}} = -\frac{\Gamma(2 - s)\Gamma\left(\frac{n}{2} + s - 1\right)}{\Gamma\left(\frac{n}{2}\right)}, \tag{5.7}$$

for n an integer. The m-series then leads to the zeta function of Riemann, and putting these remarks together we first obtain

$$2 \sum_{m=1}^{\infty} m^{2-2s} \int_0^\infty dz \, z^{2-2s} \frac{d}{dz} \left[\frac{t}{2m} + \frac{t^3(1 - 6t^2 + 5t^4)}{16m^3} \right]$$

$$= -\zeta_R(2s - 1) \frac{\Gamma(2 - s)\Gamma\left(s - \frac{1}{2}\right)}{\sqrt{\pi}}$$

$$+ \frac{1}{8} \zeta_R(1 + 2s)\Gamma(2 - s) \left\{ -\frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} + 6\frac{\Gamma\left(s + \frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} - 5\frac{\Gamma\left(s + \frac{5}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} \right\}$$

$$= -\zeta_R(2s - 1) \frac{\Gamma(2 - s)\Gamma\left(s - \frac{1}{2}\right)}{\sqrt{\pi}} - \zeta_R(1 + 2s)\frac{s(s - 1)\Gamma(2 - s)\Gamma\left(s + \frac{1}{2}\right)}{3\sqrt{\pi}}. \tag{5.8}$$

This derivation, which assumes $1 < \text{Re } s < 2$, provides the analytic continuation to $s = 0$. In the second term one needs to be a little careful as the pole at $s = 0$ coming from the $\zeta_R(1 + 2s)$ multiplies s, contributing altogether something finite. Adding up the two contributions, namely $-1/6$ and $+1/6$, the total contribution is 0.

This shows that $m = 0$ and the asymptotic terms do not contribute to order λ. The fact that formulae (5.6) and (5.8) have non-overlapping domains of validity is irrelevant, taking into account the remark made below (5.2).

Let us now consider the most difficult term (which usually one is not able to handle analytically, but here it works out fine). We want to find

$$Z^{(1)} = \int_0^\infty dx \, x^2 \frac{d}{dx} 2 \sum_{m=1}^\infty \left\{ K_m(x) I_m(x) - \frac{t}{2m} - \frac{t^3(1 - 6t^2 + 5t^4)}{16m^3} \right\}, \quad (5.9)$$

which is the last term in (5.5) at $s = 0$ with the substitution $x = zm$ and therefore here $t = 1/\sqrt{1 + (x/m)^2}$; this representation, at least for the first term, is better for the following application of the addition theorem of the Bessel functions. Remember, we are able to put $s = 0$ and interchange summation and integration because $Z^{(1)}$ is well defined by construction.

We will now show that this finite quantity actually vanishes.

As it stands, the summation cannot be performed. In order to be able to deal with the individual terms separately, we introduce an oscillatory factor. For $Z^{(1)}$ we therefore write

$$Z^{(1)} = \lim_{\varphi \to 0} \int_0^\infty dx \, x^2 \frac{d}{dx} 2 \operatorname{Re} \sum_{m=1}^\infty e^{im\varphi} \left\{ K_m(x) I_m(x) - \frac{t}{2m} - \frac{t^3(1 - 6t^2 + 5t^4)}{16m^3} \right\}. \quad (5.10)$$

The advantage is that we can now use the addition theorem for Bessel functions (4.2) to see that

$$2 \operatorname{Re} \sum_{m=1}^\infty e^{im\varphi} K_m(x) I_m(x) = K_0 \left(2x \sin \frac{\varphi}{2} \right) - K_0(z) I_0(z). \quad (5.11)$$

This shows that

$$Z^{(1)} = \lim_{\varphi \to 0} \int_0^\infty dx \, x^2 \frac{d}{dx} \left\{ K_0 \left(2x \sin \frac{\varphi}{2} \right) - K_0(x) I_0(x) \right.$$
$$\left. - \sum_{m=1}^\infty \frac{\cos(m\varphi)}{m} t - \frac{1}{8} \sum_{m=1}^\infty \frac{\cos(m\varphi)}{m^3} t^3(1 - 6t^2 + 5t^4) \right\}. \quad (5.12)$$

As they stand, the two series cannot be performed further, because t contains a nontrivial m-dependence. Of course one would like to substitute back $x = zm$, but in order to do so one needs to separate all terms and perform the x-integration for each term individually. This gives divergent results. So in order to be able to consider each term individually, we have additionally to regularize the x-integration and will therefore consider (effectively reinserting the zeta-function regularization)

$$Z_{\text{reg}}^{(1)}(s) = \lim_{\varphi \to 0} \lim_{\alpha \to 0} [Z_{11}(\alpha, \varphi) + Z_{12}(\alpha, \varphi) + Z_{13}(\alpha, \varphi) + Z_{14}(\alpha, \varphi)], \quad (5.13)$$

where

$$Z_{11}(\alpha, \varphi) = \int_0^\infty dx \, x^{2-\alpha} \frac{d}{dx} K_0 \left(2x \sin \frac{\varphi}{2} \right), \quad (5.14a)$$

$$Z_{12}(\alpha, \varphi) = - \int_0^\infty dx \, x^{2-\alpha} \frac{d}{dx} K_0(x) I_0(x), \quad (5.14b)$$

$$Z_{13}(\alpha, \varphi) = - \int_0^\infty dx \, x^{2-\alpha} \frac{d}{dx} \sum_{m=1}^\infty \frac{\cos(m\varphi)}{m} t, \quad (5.14c)$$

$$Z_{14}(\alpha, \varphi) = - \frac{1}{8} \int_0^\infty dx \, x^{2-\alpha} \frac{d}{dx} \sum_{m=1}^\infty \frac{\cos(m\varphi)}{m^3} t^3(1 - 6t^2 + 5t^4). \quad (5.14d)$$

The integral for Z_{11} can be performed for $\mathrm{Re}\,\alpha < 2$ and one obtains

$$Z_{11}(\alpha, \varphi) = -\frac{1}{2}\Gamma\left(1 - \frac{\alpha}{2}\right)\Gamma\left(2 - \frac{\alpha}{2}\right)\sin^{\alpha-2}\left(\frac{\varphi}{2}\right). \tag{5.15}$$

At $\alpha = 0$ this gives

$$Z_{11}(0, \varphi) = -\frac{1}{2}\sin^{-2}\left(\frac{\varphi}{2}\right). \tag{5.16}$$

Ultimately we want to send $\varphi \to 0$; remembering that the oscillating factor was introduced to make each separate sum convergent, it is expected that in this limit divergences might occur. By construction, this divergent piece as $\varphi \to 0$ has to be cancelled by one of the remaining contributions, since at the beginning the combination of all terms was finite.

The second contribution is calculated as in (5.6), namely for $1 < \mathrm{Re}\,\alpha < 2$,

$$Z_{12}(\alpha, \varphi) = (2 - \alpha)\frac{\Gamma\left(1 - \frac{\alpha}{2}\right)^2\Gamma\left(\frac{\alpha-1}{2}\right)}{4\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)}, \tag{5.17}$$

which vanishes at $\alpha = 0$. So this does not contribute to the energy.

Next we find (we substitute back to the case where $t = 1/\sqrt{1 + z^2}$) for $1 < \mathrm{Re}\,\alpha < 4$

$$Z_{13}(\alpha, \varphi) = -\left(\int_0^\infty \mathrm{d}z\, z^{2-\alpha}\frac{\mathrm{d}}{\mathrm{d}z}t\right)\left(\sum_{m=1}^\infty m^{1-\alpha}\cos(m\varphi)\right), \tag{5.18}$$

which at $\alpha = 0$ gives

$$Z_{13}(0, \varphi) = \frac{1}{2\sin^2\varphi/2}, \tag{5.19}$$

which exactly cancels (5.16).

Finally, we find for $-1 < \mathrm{Re}\,\alpha < 4$

$$Z_{14}(\alpha, \varphi) = -\frac{1}{8}\left(\int_0^\infty \mathrm{d}z\, z^{2-\alpha}\frac{\mathrm{d}}{\mathrm{d}z}t^3(1 - 6t^2 + 5t^4)\right)\left(\sum_{m=1}^\infty \frac{\cos(m\varphi)}{m^{1+\alpha}}\right). \tag{5.20}$$

Looking at the calculation of the asymptotic terms, it is seen that the integral is the same as that encountered in the second term in (5.8) and hence at $\alpha = 0$ this vanishes. Note that this time 0 is not multiplied by any infinite quantity because $\varphi \neq 0$ at this stage. So no contribution results from this term.

Adding up all the contributions, the answer is simply $\tilde{\mathcal{E}}^{(1)} = 0$ and therefore

$$\mathcal{E}^{(1)} = 0, \tag{5.21}$$

in agreement with the argument given at the beginning of section 4 (see also appendix A).

5.2. $\mathcal{E}^{(2)}$

The second-order contribution, fortunately, is significantly simpler. The zeta-regularized version of the second-order energy reads

$$\mathcal{E}^{(2)} = \lambda^2 \frac{a^{2s}}{16\pi a^2}(1 + s[-1 + 2\ln(2\mu)])\sum_{m=-\infty}^\infty \int_0^\infty \mathrm{d}x\, x^{2-2s}\frac{\mathrm{d}}{\mathrm{d}x}K_m^2(x)I_m^2(x). \tag{5.22}$$

As before, let us define the second-order term without the prefactor,

$$\tilde{\mathcal{E}}^{(2)} = \sum_{m=-\infty}^\infty \int_0^\infty \mathrm{d}x\, x^{2-2s}\frac{\mathrm{d}}{\mathrm{d}x}K_m^2(x)I_m^2(x). \tag{5.23}$$

As it stands, the representation is well defined in the strip $1/2 < \mathrm{Re}\, s < 1$. In that strip we interchange integral and summation using (4.4). Substituting $u = 2x \sin\varphi/2$, we obtain for (5.23)

$$\tilde{\mathcal{E}}^{(2)} = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \left(2\sin\frac{\varphi}{2}\right)^{-2+2s} \int_0^\infty \mathrm{d}u\, u^{2-2s} \frac{\mathrm{d}}{\mathrm{d}u} K_0^2(u). \tag{5.24}$$

In the relevant range, namely $1/2 < \mathrm{Re}\, s < 1$, we have the integral [(4.6)]

$$\int_0^{2\pi} \mathrm{d}\varphi \left(\sin\frac{\varphi}{2}\right)^{-2+2s} = \frac{2\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}. \tag{5.25}$$

So the analytic continuation of the integral to $s = 0$ vanishes. Given that the u-integral is well behaved about $s = 0$ (equalling -1 there), we have already found the final answer to be $\tilde{\mathcal{E}}^{(2)} = 0$ and thus

$$\mathcal{E}^{(2)} = 0. \tag{5.26}$$

The first two orders of the weak-coupling expansion indeed do identically vanish.

6. Divergences in the total energy

In the following we are going to use heat-kernel knowledge to determine the divergence structure in the total energy. We consider a general cylinder of the type $\mathcal{C} = \mathbb{R} \times Y$, where Y is an arbitrary smooth two-dimensional region rather than merely being the disc. As a metric we have $\mathrm{d}s^2 = \mathrm{d}z^2 + \mathrm{d}Y^2$ from which we obtain that the zeta function (density) associated with the Laplacian on $\mathcal{C}$ is ($\mathrm{Re}\, s > 3/2$)

$$\zeta(s) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{d}k \sum_{\lambda_Y} (k^2 + \lambda_Y)^{-s} = \frac{1}{2\pi} \frac{\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \sum_{\lambda_Y} \lambda_Y^{1/2 - s}$$

$$= \frac{1}{2\pi} \frac{\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \zeta_Y\left(s - \frac{1}{2}\right). \tag{6.1}$$

Here, λ_Y are the eigenvalues of the Laplacian on Y and $\zeta_Y(s)$ is the zeta function associated with these eigenvalues. In the zeta-function scheme the Casimir energy is defined as

$$E_{\mathrm{Cas}} = \frac{1}{2} \mu^{2s} \zeta\left(s - \frac{1}{2}\right)\bigg|_{s=0}, \tag{6.2}$$

which, in the present setting, turns into

$$E_{\mathrm{Cas}} = \frac{1}{2} \mu^{2s} \frac{\Gamma(s-1)}{2\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)} \zeta_Y(s-1)\bigg|_{s=0}. \tag{6.3}$$

Expanding this expression about $s = 0$, one obtains

$$E_{\mathrm{Cas}} = \frac{1}{8\pi s} \zeta_Y(-1) + \frac{1}{8\pi}(\zeta_Y(-1)[2\ln(2\mu) - 1] + \zeta_Y'(-1)) + \mathcal{O}(s). \tag{6.4}$$

The contribution associated with $\zeta_Y(-1)$ can be determined solely from the heat-kernel coefficient knowledge, namely

$$\zeta_Y(-1) = -a_4, \tag{6.5}$$

in terms of the standard 4th heat-kernel coefficient. The contribution coming from $\zeta_Y'(-1)$ can in general not be determined. But as we see, at least the divergent term can be determined entirely by the heat-kernel coefficient.

The situation considered in the Casimir energy calculation is a δ-function shell along some smooth line Σ in the plane (here, a circle of radius a). The manifolds considered are the cylinder created by the region inside of the line and the region outside of the line; from the results the contribution from free Minkowski space has to be subtracted to avoid trivial volume divergences (the representation in terms of the Bessel functions already has Minkowski space contributions subtracted). The δ-function shell generates a jump in the normal derivative of the eigenfunctions; call the jump U (here, $U = \lambda/a$). The leading heat-kernel coefficients for this situation, namely for functions which are continuous across the boundary but which have a jump of the first normal derivative at the boundary, have been determined in [36]; the relevant a_4 coefficient is given in theorem 7.1, p 139 of that reference. The results there are very general; for our purpose there is exactly one term that survives, namely

$$a_4 = -\frac{1}{24\pi} \int_\Sigma \mathrm{d}l \, U^3 \tag{6.6}$$

which shows that

$$E_{\mathrm{Cas}}^{\mathrm{div}} = \frac{1}{192\pi^2 s} \int_\Sigma \mathrm{d}l \, U^3. \tag{6.7}$$

So no matter along which line the δ-function shell is concentrated, the first two orders in a weak-coupling expansion do not contribute any divergences in the total energy. But the third order does, and the divergence is given above.

For the example considered, as mentioned, $U = \lambda/a$ is constant, and the integration leads to the length of the line which is $2\pi a$. Thus, we get for this particular example

$$\mathcal{E}_{\mathrm{Cas}}^{\mathrm{div}} = \frac{1}{96\pi s} \frac{\lambda^3}{a^2}. \tag{6.8}$$

This can be easily checked from the explicit representation we have for the energy. We have already seen that the first two orders in λ identically vanish, while the part of the third order that potentially contributes a divergent piece is

$$-\frac{1}{8\pi a^2} \sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}x \, x^{2-2s} \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{3} \lambda^3 K_m^3(x) I_m^3(x). \tag{6.9}$$

The $m = 0$ contribution is well behaved about $s = 0$; while for the remaining sum using

$$K_m^3(mz) I_m^3(mz) \sim \frac{1}{8m^3} \frac{1}{(1+z^2)^{3/2}}, \tag{6.10}$$

we see that the leading contribution is from (5.7)

$$-\frac{\lambda^3}{12\pi a^2} \sum_{m=1}^{\infty} m^{2-2s} \int_0^\infty \mathrm{d}z \, z^{2-2s} \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{8m^3} \frac{1}{(1+z^2)^{3/2}}$$

$$= -\frac{\lambda^3}{96\pi a^2} \zeta_R(1+2s) \int_0^\infty \mathrm{d}z \, z^{2-2s} \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{(1+z^2)^{3/2}}$$

$$= \frac{\lambda^3}{96\pi a^2} \zeta_R(1+2s) \frac{\Gamma(2-s)\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(3/2)} = \frac{\lambda^3}{96\pi a^2 s} + \mathcal{O}(s^0), \tag{6.11}$$

in perfect agreement with the heat-kernel prediction (6.8).

7. Strong coupling

The strong-coupling limit of the energy (3.5), that is, the Casimir energy of a Dirichlet cylinder,

$$\mathcal{E}^{D} = -\frac{1}{8\pi a^2} \sum_{m=-\infty}^{\infty} \int_0^\infty dx \, x^2 \frac{d}{dx} \ln I_m(x) K_m(x), \qquad (7.1)$$

was worked out to high accuracy by Gosdzinsky and Romeo [31],

$$\mathcal{E}^{D} = \frac{0.000\,614\,794\,033}{a^2}. \qquad (7.2)$$

It was later redone with less accuracy by Nesterenko and Pirozhenko [37]. (Note that the preprint version of the latter contains several internal sign errors.)

For completeness, let us sketch the evaluation here. We carry out a numerical calculation (very similar to that of [37]) in the spirit of section 4.2. We add and subtract the leading uniform asymptotic expansion (for $m = 0$ the asymptotic behaviour) as follows:

$$\begin{aligned}
\mathcal{E}^{D} = -\frac{1}{8\pi a^2}\Bigg\{ &-2\int_0^\infty dx \, x \left[\ln(2x I_0(x) K_0(x)) - \frac{1}{8}\frac{1}{1+x^2}\right] \\
&+ 2\sum_{m=1}^\infty \int_0^\infty dx \, x^2 \frac{d}{dx}\left[\ln(2x I_m(x) K_m(x)) - \ln\left(\frac{xt}{m}\right) - \frac{1}{2}\frac{r_1(t)}{m^2}\right] \\
&- 2\left(\frac{1}{2} + \sum_{m=1}^\infty\right)\int_0^\infty dx \, x^2 \frac{d}{dx}\ln 2x + 2\sum_{m=1}^\infty \int_0^\infty dx \, x^2 \frac{d}{dx}\ln xt \\
&+ \sum_{m=1}^\infty \int_0^\infty dx \, x^2 \frac{d}{dx}\left[\frac{r_1(t)}{m^2} - \frac{1}{4}\frac{1}{1+x^2}\right] + \frac{1}{4}\left(\frac{1}{2} + \sum_{m=1}^\infty\right)\int_0^\infty dx \, x^2 \frac{d}{dx}\frac{1}{1+x^2}\Bigg\}.
\end{aligned} \qquad (7.3)$$

In the first two terms we have subtracted the leading asymptotic behaviour so the resulting integrals are convergent. Those terms are restored in the fourth, fifth and sixth terms. The most divergent part of the Bessel functions is removed by the insertion of $2x$ in the corresponding integral, and its removal in the third term. (Elsewhere, such terms have been referred to as 'contact terms.') The terms involving Bessel functions are evaluated numerically, where it is observed that the asymptotic value of the summand (for large m) in the second term is $1/32m^2$. The fourth term is evaluated by writing it as

$$2\lim_{s\to 0}\sum_{m=1}^\infty m^{2-s}\int_0^\infty dz \frac{z^{1-s}}{1+z^2} = 2\zeta'(-2) = -\frac{\zeta(3)}{2\pi^2}, \qquad (7.4)$$

while the same argument, as anticipated, shows that the third 'contact' term is zero[3], while the sixth term is

$$-\frac{1}{2}\lim_{s\to 0}\left[\zeta(s) + \frac{1}{2}\right]\frac{1}{s} = \frac{1}{4}\ln 2\pi. \qquad (7.5)$$

The fifth term is elementary. The result then is

$$\begin{aligned}
\mathcal{E}^{D} &= \frac{1}{4\pi a^2}(0.010\,963 - 0.022\,7032 + 0 + 0.030\,4485 + 0.218\,75 - 0.229\,735) \\
&= \frac{1}{4\pi a^2}(0.007\,724) = \frac{0.000\,6146}{a^2}, \qquad (7.6)
\end{aligned}$$

which agrees with (7.2) to the fourth significant figure.

---

[3] This argument is a bit suspect, since the analytic continuation that defines the integrals has no common region of existence. Thus, the argument in the following subsection may be preferable.

7.1. Exponential regulator

As in section 4, it may seem more satisfactory to insert an exponential regulator rather than use analytic regularization. Now it is the third, fourth and sixth terms in (7.3) that must be treated. The latter is just the negative of (4.14). We can combine the third and fourth terms to give using (4.13)

$$-\frac{1}{\delta^2} - \frac{2}{\delta^2} \int_0^\infty \frac{\mathrm{d}z\, z^3}{z^2 + \delta^2} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \frac{1}{\mathrm{e}^z - 1}.$$
(7.7)

The latter integral may be evaluated by writing it as an integral along the entire z axis, and closing the contour in the upper half plane, thereby encircling the poles at $\mathrm{i}\delta$ and at $2\mathrm{i}n\pi$, where n is a positive integer. The residue theorem then gives for that integral

$$-\frac{2\pi}{\delta^3} - \frac{\zeta(3)}{2\pi^2},$$
(7.8)

so once again we obtain the same finite part as in (7.4). In this way of proceeding, then, in addition to the finite part in (7.6), we obtain divergent terms

$$\mathcal{E}_{\mathrm{div}}^{\mathrm{D}} = \frac{1}{64a^2\delta} + \frac{1}{8\pi a^2\delta^2} + \frac{1}{4a^2\delta^3},$$
(7.9)

which, with the previous interpretation for δ, implies divergent terms in the energy proportional to L/a (shape), L (length) and aL (area), respectively. Such terms presumably are to be subsumed in a renormalization of parameters in the model. Had a logarithmic divergence occurred (as does occur in weak coupling in $\mathcal{O}(\lambda^3)$) such a renormalization would apparently be impossible—however, see section 9.

8. Local energy density

We compute the energy density from the stress tensor (3.1) or

$$\langle T^{00} \rangle = \frac{1}{2\mathrm{i}}(\partial^0 \partial^{0\prime} + \nabla \cdot \nabla')G(x, x')\bigg|_{x'=x} - \frac{\xi}{\mathrm{i}}\nabla^2 G(x, x)$$

$$= \frac{1}{16\pi^3\mathrm{i}} \int_{-\infty}^\infty \mathrm{d}k \int_{-\infty}^\infty \mathrm{d}\omega \sum_{m=-\infty}^\infty \left[\left(\omega^2 + k^2 + \frac{m^2}{r^2} + \partial_r \partial_{r'}\right) g(r, r')\bigg|_{r'=r} \right.$$

$$\left. - 2\xi \frac{1}{r} \partial_r r \partial_r g(r, r) \right].$$
(8.1)

We omit the free part of the Green's function, since that corresponds to the energy that would be present in the vacuum in the absence of the cylinder. When we insert the remainder of the Green's function (2.9b), we obtain the following expression for the energy density outside the cylindrical shell:

$$u(r) = \langle T^{00} - T_{(0)}^{00} \rangle = -\frac{\lambda}{16\pi^3} \int_{-\infty}^\infty \mathrm{d}\zeta \int_{-\infty}^\infty \mathrm{d}k \sum_{m=-\infty}^\infty \frac{I_m^2(\kappa a)}{1 + \lambda I_m(\kappa a) K_m(\kappa a)}$$

$$\times \left[\left(2\omega^2 + \kappa^2 + \frac{m^2}{r^2}\right) K_m^2(\kappa r) + \kappa^2 K_m'^2(\kappa r) - 2\xi \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} K_m^2(\kappa r) \right], \quad r > a.$$
(8.2)

The last factor in square brackets can be easily seen to be, from the modified Bessel equation,

$$2\omega^2 K_m^2(\kappa r) + \frac{1 - 4\xi}{2}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} K_m^2(\kappa r).$$
(8.3)

For the interior region, $r < a$, we have the corresponding expression for the energy density with $I_m \leftrightarrow K_m$.

8.1. Total and surface energy

We first need to verify that we recover the expression for the energy found in section 3. So let us integrate expression (8.2) over the region exterior of the cylinder and the corresponding interior expression over the inside region. The second term in (8.3) is a total derivative, while the first is exactly the one evaluated in section 3. The result is

$$
2\pi \int_0^\infty \mathrm{d}r\, ru(r) = -\frac{1}{8\pi a^2} \sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}x\, x^2 \frac{\mathrm{d}}{\mathrm{d}x} \ln[1 + \lambda I_m(x) K_m(x)]
$$

$$
- (1 - 4\xi)\frac{\lambda}{4\pi a^2} \int_0^\infty \mathrm{d}x\, x \sum_{m=-\infty}^{\infty} \frac{I_m(x) K_m(x)}{1 + \lambda I_m(x) K_m(x)}.
\tag{8.4}
$$

The first term is the total energy (3.5), but what do we make of the second term? In strong coupling, it would represent a constant that should have no physical significance (a contact term—it is independent of a if we revert to the physical variable κ as the integration variable). In general, however, there is another contribution to the total energy, residing precisely on the singular surface. This surface energy is given in general by [12–15, 34]

$$
\mathfrak{E} = -\frac{1 - 4\xi}{2\mathrm{i}} \oint_S \mathrm{d}\mathbf{S} \cdot \nabla G(x, x') \Big|_{x'=x},
\tag{8.5}
$$

where the normal to the surface is out of the region in question. In this case, it is easy to see that $\mathfrak{E}$ exactly equals the negative of the second term in (8.4). This is an example of the general theorem

$$
\int (\mathrm{d}\mathbf{r}) u(\mathbf{r}) + \mathfrak{E} = E,
\tag{8.6}
$$

that is, the total energy E is the sum of the integrated local energy density and the surface energy. The generalization of this theorem, equations (8.5) and (8.6), to curved space is given in [38]. A consequence of this theorem is that the total energy, unlike the local energy density, is independent of the conformal parameter ξ. For more on the surface term, see section 9.2 and appendix B.

8.2. Surface divergences

We now turn to an examination of the behaviour of the local energy density (8.2) as r approaches a from outside the cylinder. To do this we use the uniform asymptotic expansion (4.7a), (4.7b), where now we need to know that $\mathrm{d}\eta/\mathrm{d}z = 1/zt$. Let us begin by considering the strong-coupling limit, a Dirichlet cylinder. If we stop with only the leading asymptotic behaviour, we obtain the expression

$$
u(r) \sim -\frac{1}{8\pi^3} \int_0^\infty \mathrm{d}\kappa\, \kappa 2 \sum_{m=1}^\infty \mathrm{e}^{-m\chi} \left\{ \left[-\kappa^2 + (1 - 4\xi)\left(\kappa^2 + \frac{m^2}{r^2}\right) \right] \frac{\pi t}{2m} \right.
$$

$$
\left. + (1 - 4\xi)\kappa^2 \frac{\pi}{2mt} \frac{1}{z^2} \right\}, \qquad (\lambda \to \infty),
\tag{8.7}
$$

where

$$
\chi = -2 \left[\eta(z) - \eta\left(z\frac{a}{r}\right) \right],
\tag{8.8}
$$

and we have replaced the integral k and ζ by one over the polar variable κ as before. Here, we ignore the difference between r and a except in the exponent, and we now replace κ by mz/a. Close to the surface,

$$
\chi \sim \frac{2}{t} \frac{r - a}{r}, \qquad r - a \ll r,
\tag{8.9}
$$

and we carry out the sum over m according to

$$2\sum_{m=1}^{\infty} m^3 \, e^{-m\chi} \sim -2\frac{d^3}{d\chi^3}\frac{1}{\chi} = \frac{12}{\chi^4} \sim \frac{3}{4}\frac{t^4 r^4}{(r-a)^4}. \tag{8.10}$$

Then, the energy density behaves, as $r \to a+$,

$$u(r) \sim -\frac{3}{64\pi^2}\frac{1}{(r-a)^4}\int_0^{\infty} dz\, z[t^5 + t^3(1-8\xi)]$$

$$= -\frac{1}{16\pi^2}\frac{1}{(r-a)^4}(1-6\xi). \tag{8.11}$$

This is the universal surface divergence first discovered by Deutsch and Candelas [2]. It therefore occurs, with precisely the same numerical coefficient, near a Dirichlet plate [32] or a Dirichlet sphere [11]. It is utterly without physical significance and may be eliminated with the conformal choice for the parameter ξ, $\xi = 1/6$.

We will henceforth make this conformal choice. Then, the leading divergence depends upon the curvature. This was also worked out by Deutsch and Candelas [2]; for the case of a cylinder, that result is

$$u(r) \sim \frac{1}{720\pi^2}\frac{1}{r(r-a)^3}, \qquad r \to a+, \tag{8.12}$$

exactly $1/2$ that for a Dirichlet sphere of radius a [11]. Here, this result may be straightforwardly derived by keeping the $1/m$ corrections in the uniform asymptotic expansion (4.7a), (4.7b), as well as the next term in the expansion of χ,

$$\chi \sim \frac{2}{t}\frac{r-a}{r} + t\left(\frac{r-a}{r}\right)^2. \tag{8.13}$$

(Note that there is a sign error in (4.8) in [11].)

8.3. Weak coupling

Let us now expand the energy density (8.2) for small coupling,

$$u(r) = -\frac{\lambda}{16\pi^3}\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dk \sum_{m=-\infty}^{\infty} I_m^2(\kappa a) \sum_{n=0}^{\infty}(-\lambda)^n I_m^n(\kappa a) K_m^n(\kappa a)$$

$$\times \left\{\left[-\kappa^2 + (1-4\xi)\left(\kappa^2 + \frac{m^2}{r^2}\right)\right]K_m^2(\kappa r) + (1-4\xi)\kappa^2 K_m'^2(\kappa r)\right\}. \tag{8.14}$$

If we again use the leading uniform asymptotic expansions for the Bessel functions, we obtain the expression for the leading behaviour of the term of order λ^n,

$$u^{(n)}(r) \sim \frac{1}{8\pi^2 r^4}\left(-\frac{\lambda}{2}\right)^n \int_0^{\infty} dz\, z \sum_{m=1}^{\infty} m^{3-n}\, e^{-m\chi} t^{n-1}(t^2 + 1 - 8\xi). \tag{8.15}$$

The sum on m is asymptotic to

$$\sum_{m=1}^{\infty} m^{3-n}\, e^{-m\chi} \sim (3-n)!\left(\frac{tr}{2(r-a)}\right)^{4-n}, \qquad r \to a+, \tag{8.16}$$

so the most singular behaviour of the order λ^n term is, as $r \to a+$,

$$u^{(n)}(r) \sim (-\lambda)^n \frac{(3-n)!(1-6\xi)}{96\pi^2 r^n(r-a)^{4-n}}. \tag{8.17}$$

This is exactly the result found for the weak-coupling limit for a δ-sphere [11] and for a δ-plane [34], so this is also a universal result, without physical significance. It may be made to vanish by choosing the conformal value $\xi = 1/6$.

With this conformal choice, once again we must expand to higher order. Besides the corrections noted above, in (4.7*b*), (4.7*b*) and (8.13), we also need

$$\tilde{t} \equiv t(za/r) \sim t + (t - t^3)\frac{r - a}{r}, \qquad r \to a, \tag{8.18}$$

Then, a quite simple calculation gives

$$u^{(n)} \sim (-\lambda)^n \frac{(n - 1)(n + 2)\Gamma(3 - n)}{2880\pi^2 r^{n+1}(r - a)^{3-n}}, \qquad r \to a+, \tag{8.19}$$

which is analytically continued from the region $1 \leqslant \operatorname{Re} n < 3$. Remarkably, this is exactly one-half the result found in the same weak-coupling expansion for the leading conformal divergence outside a sphere [11]. Therefore, like the strong-coupling result (8.12), this limit is universal, depending on the sum of the principal curvatures of the interface.

9. Cylindrical shell of finite thickness

In this section, we regard the shell (annulus) to have a finite thickness δ. We consider the potential

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{2a}\phi^2\sigma(r), \tag{9.1}$$

where

$$\sigma(r) = \begin{cases} 0, & r < a_-, \\ h, & a_- < r < a_+, \\ 0, & a_+ < r. \end{cases} \tag{9.2}$$

Here $a_\pm = a \pm \delta/2$, and we set $h\delta = 1$. In the limit as $\delta \to 0$ we recover the δ-function potential considered in the rest of this paper.

As in [11], it is straightforward to find the Green's function for this potential. In fact, the result may be obtained from the reduced Green's function given there by an evident substitution. Here, we content ourselves by stating the result for the Green's function in the region of the annulus, $a_- < r, r' < a_+$:

$$
\begin{aligned}
g_m(r, r') = {}& I_m(\kappa'r_<)K_m(\kappa'r_>) + AI_m(\kappa'r)I_m(\kappa'r') \\
& + B[I_m(\kappa'r)K_m(\kappa'r') + K_m(\kappa'r)I_m(\kappa'r')] + CK_m(\kappa'r)K_m(\kappa'r'),
\end{aligned} \tag{9.3}
$$

where $\kappa' = \sqrt{\kappa^2 + \lambda h/a}$. The coefficients appearing here are

$$
\begin{aligned}
A = {}& -\frac{1}{\Xi}[\kappa I'_m(\kappa a_-)K_m(\kappa'a_-) - \kappa'I_m(\kappa a_-)K'_m(\kappa'a_-)] \\
& \times [\kappa K'_m(\kappa a_+)K_m(\kappa'a_+) - \kappa'K_m(\kappa a_+)K'_m(\kappa'a_+)],
\end{aligned} \tag{9.4a}
$$

$$
\begin{aligned}
B = {}& \frac{1}{\Xi}[\kappa I'_m(\kappa a_-)I_m(\kappa'a_-) - \kappa'I_m(\kappa a_-)I'_m(\kappa'a_-)] \\
& \times [\kappa K'_m(\kappa a_+)K_m(\kappa'a_+) - \kappa'K_m(\kappa a_+)K'_m(\kappa'a_+)],
\end{aligned} \tag{9.4b}
$$

$$
\begin{aligned}
C = {}& -\frac{1}{\Xi}[\kappa I'_m(\kappa a_-)I_m(\kappa'a_-) - \kappa'I_m(\kappa a_-)I'_m(\kappa'a_-)] \\
& \times [\kappa K'_m(\kappa a_+)I_m(\kappa'a_+) - \kappa'K_m(\kappa a_+)I'_m(\kappa'a_+)],
\end{aligned} \tag{9.4c}
$$

where the denominator is

$$
\begin{aligned}
\Xi = {}& [\kappa I_m'(\kappa a_-) K_m(\kappa' a_-) - \kappa' I_m(\kappa a_-) K_m'(\kappa' a_-)] \\
& \times [\kappa K_m'(\kappa a_+) I_m(\kappa' a_+) - \kappa' K_m(\kappa a_+) I_m'(\kappa' a_+)] \\
& - [\kappa I_m'(\kappa a_-) I_m(\kappa' a_-) - \kappa' I_m(\kappa a_-) I_m'(\kappa' a_-)] \\
& \times [\kappa K_m'(\kappa a_+) K_m(\kappa' a_+) - \kappa' K_m(\kappa a_+) K_m'(\kappa' a_+)].
\end{aligned}
\tag{9.5}
$$

The general expression for the energy density within the shell is given in terms of these coefficients by

$$
\begin{aligned}
u = {}& \frac{1}{8\pi^2} \int_0^\infty \mathrm{d}\kappa\, \kappa \sum_{m=-\infty}^\infty \left\{ \left[-\frac{\kappa^2}{2} + \frac{m^2}{r^2} + k^2 + \frac{\lambda h}{a} - 4\xi\kappa'^2 \left(1 + \frac{m^2}{\kappa'^2 r^2} \right) \right] \right. \\
& \times \left[A I_m^2(\kappa' r) + C K_m^2(\kappa' r) + 2 B K_m(\kappa' r) I_m(\kappa' r) \right] \\
& \left. + \kappa'^2(1 - 4\xi) \left[A I_m'^2(\kappa' r) + C K_m'^2(\kappa' r) + 2 B K_m'(\kappa' r) I_m'(\kappa' r) \right] \right\} \\
= {}& \frac{1}{8\pi^2} \int_0^\infty \mathrm{d}\kappa\, \kappa \left[-\kappa^2 + \frac{1 - 4\xi}{2} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \right] \\
& \times \sum_{m=-\infty}^\infty \left[A I_m^2(\kappa' r) + C K_m^2(\kappa' r) + 2 B K_m(\kappa' r) I_m(\kappa' r) \right].
\end{aligned}
\tag{9.6}
$$

9.1. Leading surface divergence

The above expressions are somewhat formidable. Therefore, to isolate the most divergent structure, we replace the Bessel functions by the leading uniform asymptotic behaviour (4.7). A simple calculation implies

$$
A \sim \frac{t_+ - t_+'}{t_+ + t_+'} \, \mathrm{e}^{-2m\eta_+'},
\tag{9.7a}
$$

$$
B \sim \frac{t_+ - t_+'}{t_+ + t_+'} \frac{t_- - t_-'}{t_- + t_-'} \, \mathrm{e}^{2m(\eta_-' - \eta_+')},
\tag{9.7b}
$$

$$
C \sim \frac{t_- - t_-'}{t_- + t_-'} \, \mathrm{e}^{2m\eta_-'},
\tag{9.7c}
$$

using the notation $t_+ = t(z_+)$, $\eta_-' = \eta(z_-')$, etc, where for example $z_-' = \kappa' a_-/m$. If we now insert this approximation into the form (9.6) for the energy density, we find

$$
\begin{aligned}
u = \langle T^{00} \rangle = {}& \frac{1}{8\pi^2 a_+^4} 2 \sum_{m=1}^\infty m \int_0^\infty \mathrm{d}z_+ z_+ t_r' \left\{ \left[\frac{t_+ - t_+'}{t_+ + t_+'} \, \mathrm{e}^{2m(\eta_r' - \eta_+')} + \frac{t_- - t_-'}{t_- + t_-'} \, \mathrm{e}^{2m(-\eta_r' + \eta_-')} \right] \right. \\
& \times \left[\frac{m^2 z_+^2}{2} (1 - 8\xi) + \left(\frac{\lambda h a_+^2}{a} + \frac{m^2 a_+^2}{r^2} \right)(1 - 4\xi) \right] \\
& \left. - m^2 z_+^2 \frac{t_+ - t_+'}{t_+ + t_+'} \frac{t_- - t_-'}{t_- + t_-'} \, \mathrm{e}^{2m(\eta_-' - \eta_+')} \right\}.
\end{aligned}
\tag{9.8}
$$

If we are interested in the surface divergence as r approaches the outer radius a_+, the dominant term comes from the first exponential factor only. Because we are considering the limit $\lambda h a \ll m^2$, we have

$$
t_+' \approx t_+ \left(1 - \frac{\lambda h}{2m^2} \frac{a_+^2}{a} t_+^2 \right),
\tag{9.9}
$$

and we have

$$u \sim -\frac{\lambda h/a}{32\pi^2 a_+^2} \sum_{m=1}^{\infty} m \int_0^{\infty} dz \, zt (1 - 8\xi + t^2) \, e^{2m(\eta_r' - \eta_+')}. \tag{9.10}$$

The sum over m is carried out according to (8.16) or

$$\sum_{m=1}^{\infty} m \, e^{2m(\eta_r' - \eta_+')} \sim \left(\frac{rt_r'}{2(r - a_+)} \right)^2, \tag{9.11}$$

and the remaining integrals over z are elementary. The result is

$$u \sim \frac{\lambda h}{96\pi^2 a} \frac{1 - 6\xi}{(r - a_+)^2}, \qquad r \to a_+. \tag{9.12}$$

This is the expected universal divergence of a scalar field near a surface of discontinuity [39], which is without significance, and which may once again be eliminated by setting $\xi = 1/6$.

9.2. Surface energy

In this subsection, we want to establish that the surface energy $\mathfrak{E}$ (8.5) is the same as the integrated local energy density in the shell when the limit $\delta \to 0$ is taken. To examine this limit, we consider $\lambda h/a \gg \kappa^2$. So we apply the uniform asymptotic expansion for the Bessel functions of κ' only. We must keep the first two terms in powers of $\kappa \ll \kappa'$:

$$\Xi \sim -\kappa'^2 \frac{I_m(\kappa a_-) K_m(\kappa a_+)}{m z_-' z_+' \sqrt{t_-' t_+'}} \sinh m(\eta_-' - \eta_+') - \frac{\kappa' \kappa}{m} \left[\frac{1}{z_+'} \sqrt{\frac{t_-'}{t_+'}} I_m'(\kappa a_-) K_m(\kappa a_+) \right.$$

$$\left. - \frac{1}{z_-'} \sqrt{\frac{t_+'}{t_-'}} I_m(\kappa a_-) K_m'(\kappa a_+) \right] \cosh m(\eta_-' - \eta_+'). \tag{9.13}$$

Because we are now regarding the shell as very thin,

$$\eta_-' - \eta_+' \approx -\frac{\delta}{a} \frac{1}{t'}, \tag{9.14}$$

where

$$t' \sim \frac{1}{z'} \sim \frac{m}{\sqrt{\lambda h a}}, \tag{9.15}$$

using the Wronskian we get

$$\Xi \sim -\frac{1}{a^2} [1 + \lambda I_m(\kappa a) K_m(\kappa a)]. \tag{9.16}$$

Then, we immediately find the interior coefficients:

$$A \sim \frac{\pi}{2} \sqrt{\lambda h a} \frac{I_m(\kappa a) K_m(\kappa a)}{1 + \lambda I_m(\kappa a) K_m(\kappa a)} \, e^{-2m\eta'}, \tag{9.17a}$$

$$B \sim \frac{1}{2} \sqrt{\lambda h a} \frac{I_m(\kappa a) K_m(\kappa a)}{1 + \lambda I_m(\kappa a) K_m(\kappa a)}, \tag{9.17b}$$

$$C \sim \frac{1}{2\pi} \sqrt{\lambda h a} \frac{I_m(\kappa a) K_m(\kappa a)}{1 + \lambda I_m(\kappa a) K_m(\kappa a)} \, e^{2m\eta'}, \tag{9.17c}$$

We now insert this into (9.6) and keep only the largest terms, thereby neglecting κ^2 relative to $\lambda h/a$. This gives a leading term proportional to h, which when multiplied by the area of the annulus $2\pi a\delta$ gives for the energy in the shell

$$\mathcal{E}_{\text{ann}} = 2\pi\delta\, au \sim (1-4\xi)\frac{\lambda}{4\pi a^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}\kappa a\,\kappa a\,\frac{I_m(\kappa a)K_m(\kappa a)}{1+\lambda I_m(\kappa a)K_m(\kappa a)} = \mathfrak{E}, \qquad (9.18)$$

which is exactly the form of the surface energy given by the negative of the second term in (8.4).

In particular, note that the term in $\mathfrak{E}$ of order λ^3 is, for the conformal value $\xi = 1/6$, exactly equal to that term in the total energy $\mathcal{E}$ (3.5) (see (6.9)):

$$\mathfrak{E}^{(3)} = \mathcal{E}^{(3)}. \qquad (9.19)$$

This means that the divergence encountered in the global energy (6.8) is exactly accounted for by the divergence in the surface energy, which would seem to provide strong evidence in favour of the renormalizablity of that divergence.

10. Conclusions

In this paper, we have extended the considerations applied to a spherical geometry in [11] to a cylindrical one. Results for the general structure of the local and global energies are rather as expected, with the leading conformal divergences in the local energy density as the surface is approached being reduced from their spherical values by a factor of $1/2$. The only possibly surprising result is that the weak-coupling limit of the total Casimir energy for a δ-function cylindrical shell vanishes through order λ^2, in agreement with other perturbative results for a cylinder, as seen in table 1. We do not yet have a complete explanation of this. Schaden [40] has presented a semiclassical calculation in terms of periodic rays which sheds light on this general phenomenon. His technique closely reproduces Boyer's result for the spherical shell, while giving a vanishing result to all orders for the cylinder. Similar conclusions were reached earlier by Balian and Duplantier [41] and by Mazzitelli *et al* [42]. This cannot be the whole story, however, since Casimir energies for cylinders only vanish through second order and do not vanish for a perfectly conducting cylinder. Here, the weak-coupling Casimir energy diverges in $\mathcal{O}(\lambda^3)$, which has its origin in the singular nature of the surface energy.

Acknowledgments

Appendix A. Perturbative divergences

A conventional field theorist might be surprised that we are able to extract finite (actually zero) values for the global energy in order λ and λ^2, when the latter contributions would seem to

correspond to divergent Feynman graphs. Thus, the first-order energy is given by (T is the very large time interval characterizing the spacetime volume under consideration)

$$E^{(1)} = \frac{i}{2T} \frac{\lambda}{a} D_+(0) \int (dx)\sigma(x), \tag{A.1}$$

which involves the massless scalar propagator $D_+(x)$ at zero coordinate, corresponding to equal field points. The latter is ill-defined. For example, for our cylindrical δ-shell geometry, we would compute from this the energy per unit length

$$\mathcal{E}^{(1)} = \frac{i\lambda}{4\pi} \frac{1}{\epsilon}, \qquad \epsilon \to 0+, \tag{A.2}$$

which, although divergent, is imaginary and independent of a. Such a contribution should be irrelevant.

Less divergent, but still ambiguous, is the second-order expression

$$E^{(2)} = \frac{i}{4T} \left(\frac{\lambda}{a}\right)^2 \int (dx)(dy)\sigma(x)D_+(x-y)\sigma(y)D_+(y-x). \tag{A.3}$$

This may be evaluated by a method very similar to that employed in [32]. If we work in $D+1$ dimensions, we obtain

$$\mathcal{E}^{(2)} = -\frac{\lambda^2}{4} \frac{\Gamma\left(\frac{3-D}{2}\right)}{(2\pi)^{(D-3)/2}} \frac{1}{a^{(D+1)/2}} \int_0^\infty dx\, x^{D-2} J_0^2(x) \int_0^\infty du\, u^{(D-3)/2}(1-u)^{(D-3)/2}. \tag{A.4}$$

For $D > 1$, the second integral may be evaluated as a beta function. The integral over the squared Bessel function is, for $1 < \text{Re}\, D < 2$,

$$\int_0^\infty dx\, x^{D-2} J_0^2(x) = \frac{\Gamma\left(1-\frac{D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{2\sqrt{\pi}\,\Gamma^2\left(\frac{3-D}{2}\right)}. \tag{A.5}$$

When this is inserted into (A.4), a factor of $\Gamma((3-D)/2)$ remains in the denominator, so the second-order energy evidently vanishes in three dimensions.

We can also proceed directly in three dimensions, if we regulate the integral using a proper time method as described in [32]. If, as there, we exclude a coincident field term (contact term), we are led to

$$\mathcal{E}^{(2)} = \frac{\lambda^2\pi^2}{4} \frac{d}{d\alpha} \int_0^\infty dq\, q^{2\alpha+1} J_0^2(qa)\bigg|_{\alpha=0}. \tag{A.6}$$

We encounter the same Bessel-function integral as in (A.5), so continuing the integral from $-1 < \text{Re}\,\alpha < -1/2$ we again see that $\mathcal{E}^{(2)} = 0$. (This same continuation process gives the correct second-order energy for a sphere: $E^{(2)} = \lambda^2/32\pi a$ for a dimensionless coupling constant λ.)

In the balance of this appendix, we sketch an evaluation of $\mathcal{E}^{(1)}$ along the lines of that given for $\mathcal{E}^{(2)}$ in section 4.2. We regulate the sum and integral by inserting an exponential cutoff, $\delta \to 0+$:

$$\mathcal{E}^{(1)} = -\frac{\lambda}{4\pi a^2} \sum_{m=0}^\infty{}' \int_0^\infty dx\, x^2 \frac{d}{dx}[I_m(x)K_m(x)]\, e^{-x\delta}, \tag{A.7}$$

where the prime on the summation sign means that the $m = 0$ term is counted with half weight, which we break up into five parts,

$$\mathcal{E}^{(1)} = -\frac{\lambda}{8\pi a^2}(\text{I} + \text{II} + \text{III} + \text{IV} + \text{V}). \tag{A.8}$$

The first term is the $m = 0$ contribution, suitably subtracted to make it convergent (so the convergence factor may be omitted),

$$\text{I} = \int_0^\infty \text{d}x \, x^2 \frac{\text{d}}{\text{d}x} \left[I_0(x) K_0(x) - \frac{1}{2\sqrt{1+x^2}} \right] = -1. \tag{A.9}$$

The second term is the above subtraction,

$$\text{II} = \frac{1}{2} \int_0^\infty \text{d}x \, x^2 \left(\frac{\text{d}}{\text{d}x} \frac{1}{\sqrt{1+x^2}} \right) \text{e}^{-x\delta} \sim -\frac{1}{2\delta} + 1, \tag{A.10}$$

as may be verified by breaking the integral in two parts at Λ, $1 \ll \Lambda \ll 1/\delta$. The third term is the sum over the mth Bessel function with the two leading asymptotic approximants (5.3) subtracted:

$$\text{III} = 2 \sum_{m=1}^\infty \int_0^\infty \text{d}x \, x^2 \frac{\text{d}}{\text{d}x} \left[I_m(x) K_m(x) - \frac{t}{2m} \left(1 + \frac{t^2}{8m^2} (1 - 6t^2 + 5t^4) \right) \right] = 0. \tag{A.11}$$

Numerically, each term in the sum seems to be zero to machine accuracy. This is verified by computing the higher order terms in that expansion, in terms of the polynomials in (4.10),

$$I_m(x) K_m(x) - \frac{t}{2m} \left(1 + \frac{t^2}{8m^2} (1 - 6t^2 + 5t^4) \right) \sim \frac{t}{4m^5} \left[r_2(t) - \frac{1}{4} r_1^2(t) \right]$$

$$+ \frac{t}{4m^7} \left[r_3(t) - \frac{1}{2} r_1(t) r_2(t) + \frac{1}{8} r_1^3(t) \right] + \cdots, \tag{A.12}$$

which terms are easily seen to integrate to zero. The fourth term is the leading subtraction which appeared in the third term:

$$\text{IV} = \sum_{m=1}^\infty m \int_0^\infty \text{d}z \, z^2 \left(\frac{\text{d}}{\text{d}z} t \right) \text{e}^{-mz\delta}. \tag{A.13}$$

If we first carry out the sum on m we obtain

$$\text{IV} = -\frac{1}{4} \int_0^\infty \text{d}z \, z^3 \frac{1}{(1+z^2)^{3/2}} \frac{1}{\sinh^2 z\delta/2}$$

$$\sim -\frac{1}{\delta^2} + \frac{1}{2\delta} - \frac{1}{6}, \tag{A.14}$$

as again may be easily verified by breaking up the integral. The final term, if unregulated, is the form of infinity times zero:

$$\text{V} = \frac{1}{8} \sum_{m=1}^\infty \frac{1}{m} \int_0^\infty \text{d}z \, z^2 \frac{\text{d}}{\text{d}z} (t^3 - 6t^5 + 5t^7) \text{e}^{-mz\delta}. \tag{A.15}$$

Here, the sum on m gives

$$\sum_{m=1}^\infty \frac{1}{m} \text{e}^{-mz\delta} = -\ln(1 - \text{e}^{-z\delta}), \tag{A.16}$$

and so we can write

$$\text{V} = \frac{1}{16} \frac{\text{d}}{\text{d}\alpha} \int_0^1 \text{d}u (1-u)^\alpha u^{-2-\alpha} (u^{3/2} - 6u^{5/2} + 5u^{7/2}) \bigg|_{\alpha=0} = \frac{1}{6}. \tag{A.17}$$

Adding together (A.9), (A.10), (A.11), (A.14) and (A.17), we obtain

$$\mathcal{E}^{(1)} = \frac{\lambda}{8\pi a^2 \delta^2} + 0, \tag{A.18}$$

that is, the $1/\delta$ and constant terms cancel. The remaining divergence may be interpreted as an irrelevant constant, since $\delta = \tau/a$, with τ being regarded as a point-splitting parameter. The correspondence of the terms in this evaluation with those in section 5 is rather immediate.

Appendix B. Surface energy

Here, we suggest an alternative derivation of the surface energy term for $\xi = 0$. For the scalar Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{2a}\delta(r-a)\phi^2, \tag{B.1}$$

the response to a general coordinate transformation yields the expected stress tensor, including the interaction term with the cylindrical surface:

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi + g^{\mu\nu}\mathcal{L}. \tag{B.2}$$

The integral of the δ-function term in the energy density should be the surface energy

$$\mathfrak{E} = 2\pi\int_0^\infty \mathrm{d}r\, r\frac{\lambda}{2a}\delta(r-a)\langle\phi^2(r)\rangle = \pi\lambda\langle\phi(a)^2\rangle, \tag{B.3}$$

or, in terms of the Green's function,

$$\begin{aligned}
\mathfrak{E} &= \frac{\pi\lambda}{\mathrm{i}}\int_{-\infty}^\infty\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^\infty\frac{\mathrm{d}k}{2\pi}\frac{1}{2\pi}\sum_{m=-\infty}^\infty g_m(a,a) \\
&= -\frac{\lambda}{4\pi a^2}\int_0^\infty \mathrm{d}x\, x\sum_{m=-\infty}^\infty\frac{\lambda K_m^2(x)I_m^2(x)}{1+\lambda I_m(x)K_m(x)}.
\end{aligned} \tag{B.4}$$

This is similar to the surface energy given by (9.18) for $\xi = 0$; the difference between the two expressions is

$$-\frac{\lambda}{4\pi a^2}\sum_{m=-\infty}^\infty\int_0^\infty \mathrm{d}x\, xI_m(x)K_m(x) = -\frac{\lambda}{4\pi}\int_0^\infty \mathrm{d}\kappa\, \kappa K_0(0), \tag{B.5}$$

if we use the addition theorem (4.2) in the singular limit $\rho' \to \rho$. Such a term, a singular constant, would seem to be a physically irrelevant contact term, so the two versions of the surface energy appear to be equivalent.

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